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On a High Accuracy Finite Difference Method

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A highly accurate finite difference method is proposed for the numerical solution of partial differential equations that describe initial value problems. It includes a spatial smoothing operation and avoids the computational mode. The method is presented here for one space dimension and is tested on a linear equation, where it gives highly accurate results for the computed phase speed. The solution of a nonlinear equation confirms unconditional nonlinear stability.

1. INTRODUCTION

The numerical solution of partial differential equations requires some kind of discretization. The conventional finite difference methods used for this purpose have the property that the field must vary slowly over one grid length to be predicted with good accuracy. For a given problem, one can easily improve accuracy by using a smaller grid length. However, the numerical error does not decrease very quickly with decreasing grid length (see for example diagrams in [1]). On the other hand, a shortening of the grid length requires much more computation time.

In this paper another method of improving accuracy is considered. We associate several localized basis functions with every gridpoint. The accuracy achieved depends on the number of basis functions per gridpoint. In the present paper, the simplest cases of this concept are described in some detail. The presentation is confined to one space dimension and to the cases with one or two basis functions per gridpoint. These two cases will be referred to as the first-degree and second-degree method, respectively. The general case will be given in a paper to be published later.

The first-degree method is a special form of the Lax–Friedrichs method. The second-degree method shows a new technical feature. At a fixed time the fields are characterized by two kinds of constants, the field values at the gridpoints x_ν and the second derivatives at the intermediate points $x_{\nu+(1/2)}$. These constants can be chosen independently to describe initial values, and there is a method of computing both kinds of constants at later times.

The characteristic properties of the methods presented can be summarized as follows:

(a) Between the gridpoints the field is represented as a linear combination of basis functions.

(b) Time translation is done by a simple time step.

(c) There is a spatial smoothing operation.

(d) The principle of locality is satisfied. The last property means that in one time step each gridpoint is influenced only by its nearest neighbors.

Other methods satisfying condition (a) are the truncated Fourier expansion scheme [2], the finite element scheme [3], and the cubic spline method [4, 5]. Often regularity properties of the basis functions are considered to be essential. The truncated Fourier expansion scheme uses analytic basis functions and in the cubic spline method they are two times contiuously differenciable. All finite element schemes mentioned in [3] use basis functions that are at least once differenciable. In the present paper, we only require that the fields be continuous.

In comparison with other finite difference schemes, the time step is performed in a rather unusual way. For example, in [6] and [2] many finite difference schemes were reviewed, and in all of these schemes the information on the field $\phi(x, t)$ is given by the gridpoint values of ϕ. Consequently, the time translation is done only at the gridpoints. This involves the Taylor coefficients $(\partial/\partial t)\, \phi(x, t)$, and perhaps $(1/2)(\partial^2/\partial t^2)\, \phi(x, t)$ at the chosen points. The fair representation of the spatial derivatives of the field at the new time level must be a consequence of the chosen spatial interpolation procedure.

However, in the second-degree method, the time step consists of two parts. First, the time translating step is done by explicit evaluation of the Taylor coefficients $(\partial/\partial t)\, \phi(x_{\nu+(1/2)}, t)$, $(\partial^2/2\partial t^2)\, \phi(x_{\nu+(1/2)}, t)$ and $(\partial^2/(\partial x\, \partial t))\, \phi(x_{\nu+(1/2)}, t)$. This procedure will leave the field with discontinuities at the gridpoints x_ν. Therefore, a spatial smoothing operation is applied to reach the old continuous form of the field representation.

Property (b) means that the computational mode, which is present in the leapfrog method, is avoided. According to [6, 7], one can expect good nonlinear stability properties from (b) and (c).

Many ordinary finite difference schemes are in accordance with the principle of locality. However, it is often sacrificed by more advanced schemes, such as higher-order [8, 9], truncated Fourier series, finite element, or cubic spline schemes.

When using localized basis functions, it seems reasonable to impose the principle of locality because the Courant–Levy condition states that the fastest possible waves can only reach the nearest neighboring points in one time step. In the present paper the condition of locality leads to a relatively simple formalism and keeps the necessary computing time limited.

The method is presented in Section 2. In Section 3, the accuracy of wave solutions to the linear advection equation is considered. Section 4 gives further computational evidence, including a nonlinear example to test the nonlinear stability properties of the second-degree scheme.

2. Statement of the Method

The method presented here is applicable to differential equations of the form

$$(\partial/\partial t)\mathbf{r} = F(\mathbf{r}, (\partial/\partial x)\mathbf{r}, x) \tag{1}$$

for the first-degree method, and

$$(\partial/\partial t)\mathbf{r} = F(\mathbf{r}, (\partial/\partial x)\mathbf{r}, (\partial^2/\partial x^2)\mathbf{r}, x)$$

for the second-degree method. The vector $\mathbf{r}$ is a function of x and t. The function F must be an analytic function of its arguments. In most cases of practical importance F is actually a polynomial.

For the sake of simplicity the description of the method will be given for the case with a periodic boundary condition

$$\mathbf{r}(x, t) = \mathbf{r}(x + L, t).$$

Definition of Finite Dimensional Function Spaces

On the interval $[0, L]$ we assume $N + 1$ gridpoints $x_0, \ldots, x_N$ with

$$x_0 = 0, \qquad x_N = L$$

and constant distances

$$\Delta x = x_{\nu+1} - x_\nu .$$

We will also use the intermediate points

$$x_{\nu+(1/2)} = (x_{\nu+1} + x_\nu)/2.$$

We further define three functions:

$$f_1(y) = 1, \qquad f_2(y) = y, \qquad f_3(y) = (1/2)(y^2 - (\Delta x^2/4))$$

for $|y| \leqslant \Delta x/2$.

The finite dimensional function spaces S_ρ and P_ρ are needed, where S_ρ contains continuous functions and P_ρ contains functions that may have discontinuities at the gridpoints x_ν. The index ρ refers to the degree of the method. We consider only the cases $\rho = 1$ and $\rho = 2$.

In order to define an element of the set S_2, let the numbers $\phi_0, \ldots, \phi_N$, $\phi_{xx,(1/2)}, \ldots, \phi_{xx,N-(1/2)}$ be given, with

$$\phi_0 = \phi_N . \tag{2}$$

The corresponding function $\phi(x)$ of S_2 is then defined by parabola pieces for each interval $[x_\nu, x_{\nu+1}]$:

$$\begin{aligned}\phi(x) = \frac{\phi_{\nu+1} + \phi_\nu}{2} f_1(x - x_{\nu+(1/2)}) + \frac{\phi_{\nu+1} - \phi_\nu}{\Delta x} f_2(x - x_{\nu+(1/2)}) \\ + \phi_{xx,\nu+(1/2)} f_3(x - x_{\nu+(1/2)})\end{aligned} \tag{3}$$

for

$$| x - x_{\nu+(1/2)} | < \Delta x/2$$

which chain continuously together as a consequence of $f_3(\pm\Delta x/2) = 0$ and $f_2(\pm\Delta x/2) = \pm(\Delta x/2)$. The derivatives $\partial\phi/\partial x$, $\partial^2\phi/\partial x^2$ may be formed from (3), especially at the intermediate points $x_{\nu+(1/2)}$. These are needed for the execution of a time step, described later, from which new quantities $\bar{\phi}_{\nu+(1/2)}$, $\bar{\phi}_{x,\nu+(1/2)}$, $\bar{\phi}_{xx,\nu+(1/2)}$, $\nu \in \{0, \ldots, N-1\}$ will follow. These quantities are then used to define a member $\bar{\phi}(x)$ of the function space P_2 having a higher dimension than S_2:

$$\bar{\phi}(x) = \bar{\phi}_{\nu+(1/2)} + \bar{\phi}_{x,\nu+(1/2)}(x - x_{\nu+(1/2)}) + (1/2)\,\bar{\phi}_{xx,\nu+(1/2)}(x - x_{\nu+(1/2)})^2 \tag{4}$$

for $| x - x_{\nu+(1/2)} | < \Delta x/2$.

$\bar{\phi}(x)$ is composed of parabola pieces in the same intervals $[x_\nu, x_{\nu+1}]$. In general, $\bar{\phi}(x)$ will have unremovable jump discontinuities at the gridpoints x_ν. The elements of S_1 and P_1 are those members of S_2 and P_2, respectively, that have the property

$$\phi_{xx,(1/2)} = 0, \ldots, \phi_{xx,N-(1/2)} = 0$$

or

$$\bar{\phi}_{xx,(1/2)} = 0, \ldots, \bar{\phi}_{xx,N-(1/2)} = 0.$$

In the following, we also consider the shifted grid, with gridpoints $x_{\nu+(1/2)}$ and grid intervals centered at the points x_ν. The two grids are shown in Fig. 1. In a way analogous to the definition of S_ρ and P_ρ we define function spaces S_ρ' and P_ρ' on the shifted grid.

The corresponding constants are denoted by

$$\phi'_{1/2}, \ldots, \phi'_{N+(1/2)}, \qquad \phi'_{xx,1}, \ldots, \phi'_{xx,N}$$

$$\bar{\phi}_1', \ldots, \bar{\phi}_N', \qquad \bar{\phi}'_{x,1}, \ldots, \bar{\phi}'_{x,N}, \qquad \bar{\phi}'_{xx,1}, \ldots, \bar{\phi}'_{xx,N}.$$

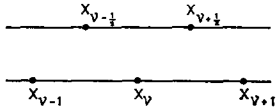

FIG. 1. The original and the shifted grid.

The periodic boundary condition requires $\phi'_{(N+1/2)} = \phi'_{(1/2)}$. For the definition of the corresponding functions $\phi'(X)$ and $\bar{\phi}'(X)$ from P_ρ' and S_ρ', respectively, which are composed of parabola pieces in the intervals $[x_{\nu-(1/2)}, x_{\nu+(1/2)}]$, one can use formulas (3) and (4) with half-integer values for ν and all functions and constants primed.

Grid Shifting

Our aim is now to approximate functions of P_ρ by functions from S_ρ. We have in mind the principle of locality, and consequently, look for such a method in which the approximating function at a certain point depends only on the values of the given function at a distance of at most Δx from the point.

In the first-degree case, let a function $\bar{\phi}(x) \in P_1$ be given. For the approximating function $\phi'(x) \in S_1'$, we define the constants $\phi'_{(1/2)},..., \phi'_{N+(1/2)}$ by

$$\phi'_{\nu+(1/2)} = \bar{\phi}_{\nu+(1/2)} \tag{5}$$

for $\nu \in \{0,..., N-1\}$ and $\phi'_{N+(1/2)} = \phi'_{(1/2)}$.

In the second-degree case, a function $\bar{\phi}(x) \in P_2$ with corresponding constants $\bar{\phi}_{(1/2)},..., \bar{\phi}_{N-(1/2)}$, $\bar{\phi}_{x,(1/2)},..., \bar{\phi}_{x,N-(1/2)}$ and $\bar{\phi}_{xx,(1/2)},..., \bar{\phi}_{xx,N-(1/2)}$ is given. We must define the numbers $\phi'_{(1/2)},..., \phi'_{N+(1/2)}$ and $\phi'_{xx,1},..., \phi'_{xx,N}$ corresponding to a function $\phi'(x) \in S_2'$. The $\phi'_{\nu+(1/2)}$ are again defined by Eq. (5). The $\phi'_{xx,\nu}$ are determined by minimizing the integral $\int_{x_{\nu-(1/2)}}^{x_{\nu+(1/2)}} (\phi'(x) - \bar{\phi}(x))^2 \, dx$. This leads to the condition

$$\frac{\partial}{\partial \phi'_{xx,\nu}} \int_{-\Delta x/2}^{\Delta x/2} dx \left(\frac{\phi'_{\nu+(1/2)} + \phi'_{\nu-(1/2)}}{2} f_1(x) + \frac{\phi'_{\nu+(1/2)} - \phi'_{\nu-(1/2)}}{\Delta x} f_2(x) \right.$$
$$\left. + \phi'_{xx,\nu} f_3(x) - \bar{\phi}(x - x_\nu) \right)^2 = 0$$

or

$$\phi'_{xx,\nu} \int_{-\Delta x/2}^{\Delta x/2} (f_3(x))^2 \, dx - \bar{\phi}_{x,\nu-(1/2)} \int_{-\Delta x/2}^{0} f_3(x) \left(x + \frac{\Delta x}{2} \right) dx$$
$$- \bar{\phi}_{xx,\nu-(1/2)} \int_{-\Delta x/2}^{0} f_3(x) \frac{(x + (\Delta x/2))^2}{2} \, dx - \bar{\phi}_{x,\nu+(1/2)} \int_{0}^{\Delta x/2} f_3(x) \left(x - \frac{\Delta x}{2} \right) dx$$
$$- \phi_{xx,\nu+(1/2)} \int_{0}^{\Delta x/2} f_3(x) \frac{(x - (\Delta x/2))^2}{2} \, dx = 0.$$

For each value of ν we get a linear equation in the single unknown $\phi'_{xx,\nu}$ and obtain

$$\phi'_{xx,\nu} = -\frac{9}{16}\,\frac{\bar{\phi}_{xx,\nu+(1/2)} + \bar{\phi}_{xx,\nu-(1/2)}}{2} + \frac{25}{16}\,\frac{\bar{\phi}_{x,\nu+(1/2)} - \bar{\phi}_{x,\nu-(1/2)}}{\Delta x}. \tag{6}$$

For the computation of $\phi'_{xx,N}$ one uses Eq. (6) in combination with

$$\bar{\phi}_{xx,N+(1/2)} = \bar{\phi}_{xx,(1/2)}, \qquad \bar{\phi}_{x,N+(1/2)} = \bar{\phi}_{x,(1/2)}$$

(see Fig. 2).

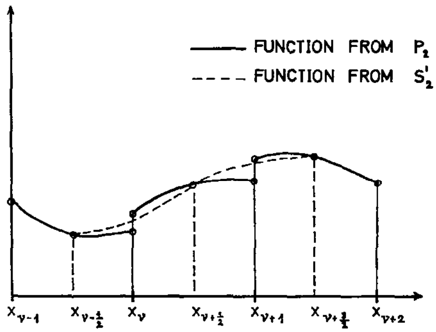

FIG. 2. Approximating functions from P_2 by functions from S_2.

Thus, we have constructed a mapping Q from P_ρ to S_ρ'. In a similar way, we get a mapping Q' from P_ρ' to S_ρ. The transformation equations belonging to Q' can be obtained from Eqs. (6) and (5) by making the unprimed constants primed and vice versa, and using these equations with half-integer ν. The grid shifting operations Q and Q', which are performed every time step, are the spatial smoothing operations of the method.

Time Translation

For the time development we use the equation

$$\mathbf{r}(x, t + \Delta t) = \mathbf{r}(x, t) + \mathbf{r}_t(x, t)\,\Delta t + \mathbf{r}_{tt}(x, t)(\Delta t^2/2). \tag{7}$$

For the first-degree method we put

$$\mathbf{r}_{tt}(x, t) = 0.$$

To determine the vector functions $\mathbf{r}_t(x, t)$ and $\mathbf{r}_{tt}(x, t)$, we insert Eq. (7) into Eq. (1):

$$\begin{aligned}&(\partial/\partial(t + \Delta t))\,\mathbf{r}(x, t + \Delta t)\\ &\quad = F(\mathbf{r}(x, t + \Delta t), (\partial/\partial x)\,\mathbf{r}(x, t + \Delta t), (\partial^2/\partial x^2)\,\mathbf{r}(x, t + \Delta t), x)\end{aligned} \tag{8}$$

and expand in a power series in Δt. By comparing the coefficients of the different Δt^{μ}, one can determine $\mathbf{r}_t(x, t)$ and $\mathbf{r}_{tt}(x, t)$.

If, for a fixed value of t, the components of $\mathbf{r}$ are elements of S_ρ or S_ρ', Eq. (7) defines functions belonging to P_ρ or P_ρ', respectively. In nonlinear cases one must neglect terms of sufficiently high order in order to get functions belonging to P_ρ or P_ρ'.

Thus, the time translation T maps S_ρ to P_ρ or S_ρ' to P_ρ'. The time translation method proposed in [10] is somewhat similar to this method.

Numerical Procedure

A time step is just a time translation mapping T followed by one of the grid shiftings Q or Q'. For example, one obtains in this way a function belonging to S_ρ' from a function in S_ρ. In the next time step one again obtains a function of S_ρ.

$$\begin{tikzcd} S_\rho \arrow[r, "T"'] & P_\rho \arrow[d, "Q'"'] \\ P_\rho' \arrow[u, "Q"] & S_\rho' \arrow[l, "T"] \end{tikzcd}$$

It is convenient to perform the computation in three steps. They will be described here for the second-degree case. The first-degree case can be obtained by dropping the second-order terms in the following equations.

We consider the case that the fields are given as functions belonging to S_2 at time level n. For the time level $n + 1$ they must be represented as functions belonging to S_2'. The other case can be obtained from the following formula by using half-integer subscripts and making primed constants unprimed and vice versa.

First, the interpolation step is performed. This will give all spatial derivatives of a field $\phi^n(x)$ at $x_{\nu+(1/2)}$. $\phi^n_{xx,\nu+(1/2)}$ is already given. The other derivatives are computed according to Eq. (3):

$$\begin{aligned} \phi^n_{\nu+(1/2)} &= (\phi^n_{\nu+1} + \phi_\nu{}^n)/2 - (\Delta x^2/8)\, \phi^n_{xx,\nu+(1/2)} \\ \phi^n_{x,\nu+(1/2)} &= (\phi^n_{\nu+1} - \phi_\nu{}^n)/\Delta x. \end{aligned}$$

The time translation step will then yield a function $\bar{\phi}^{n+1}(x) \in P_2$. Only this step depends on the equations to be solved. In Eqs. (9), (11), and (13), the space index has the constant value $\nu + (1/2)$ and will be omitted. For every field $\phi(X, t)$, Eq. (7) is implemented by

$$\begin{aligned} \bar{\phi}^{n+1} &= \phi^n + \phi_t{}^n\, \Delta t + \phi^n_{tt}(\Delta t^2/2) \\ \bar{\phi}^{n+1}_x &= \phi_x{}^n + \phi^n_{tx}\, \Delta t \\ \bar{\phi}^{n+1}_{xx} &= \phi^n_{xx}. \end{aligned} \tag{9}$$

The missing coefficients in Eq. (9) will be given for two examples:

(a) For the linear advection equation

$$(\partial/\partial t)\phi = (\partial/\partial x)\phi \tag{10}$$

one obtains

$$\phi_t{}^n = \phi_x{}^n \qquad \phi_{tx}^n = \phi_{xx}^n \qquad \phi_{tt}^n = \phi_{tx}^n . \tag{11}$$

(b) For the nonlinear equations of a barotropic fluid

$$\frac{\partial}{\partial t} U = -U \frac{\partial}{\partial x} U - \frac{\partial}{\partial x} H \qquad \frac{\partial}{\partial t} H = -\frac{\partial}{\partial x} (H \cdot U) \tag{12}$$

one obtains

$$\begin{aligned}
U_t{}^n &= -U^n U_x{}^n - H_x{}^n \\
U_{tx}^n &= -U^n U_{xx} - U_x{}^n U_x{}^n - H_{xx}^n \\
U_{tt}^n &= -U_t{}^n U_x{}^n - U^n U_{tx}^n - H_{tx}^n \\
H_t{}^n &= -(H_x{}^n U^n + H^n U_x{}^n) \\
H_{tx}^n &= -(H_{xx}^n U^n + H^n U_{xx}^n + 2H_x{}^n U_x{}^n) \\
H_{tt}^n &= -(H_t{}^n U_x{}^n + H_{tx}^n U^n + H^n U_{tx}^n + H_x{}^n U_t{}^n).
\end{aligned} \tag{13}$$

It is possible to evaluate terms like ϕ_{txx}^n in Eqs. (9). These terms of third and higher order are neglected in the present paper. For the linear case (Eq. (10)) they are actually zero.

Lastly, the spatial smoothing step can be done using Eqs. (5) and (6). The right-hand sides of these equations are determined by Eqs. (9).

3. Wave Solutions of the Advection Equations

We choose $\Delta x = 1$. The initial values are given in the first-degree case by

$$\phi_\nu{}^0 = Ae^{i(2\pi/L)\nu}$$

and in the second-degree case by

$$\phi_\nu{}^0 = Ae^{i(2\pi/L)\nu} \qquad \phi_{xx,\nu+(1/2)}^0 = Be^{i(2\pi/L)\nu}. \tag{14}$$

The wave solution of Eq. (10) is

$$\phi^n(x, t) = A^n e^{i(2\pi/L)(x+t)}. \tag{15}$$

For a given $A = A''$, Eqs. (14) and (3) will approximate Eq. (15) at $t = 0$ only for a special choice of B. The general combination of A and B will specify the initial values for a wave solution (Eq. (15)) plus a small scale feature that will be smoothed out quickly. After $2n$ time steps, one has in the first and second-degree cases,

$$\phi_\nu^{2n} = U^{2n}\phi_\nu^{\,0} \tag{16}$$

or

$$\begin{pmatrix} \phi_\nu^{2n} \\ \phi_{xx,\nu+(1/2)}^{2n} \end{pmatrix} = V^{2n} \begin{pmatrix} \phi_\nu^{\,0} \\ \phi_{xx,\nu+(1/2)}^{0} \end{pmatrix}, \tag{17}$$

respectively, where

$$U = \cos\left(\frac{\pi}{L}\right) + 2i\,\Delta t \sin\left(\frac{\pi}{L}\right) \tag{18}$$

and

$$V = \begin{pmatrix} \cos\left(\frac{\pi}{L}\right) + 2\Delta t i \sin\left(\frac{\pi}{L}\right), & -\frac{1}{2}(0.25 - \Delta t^2)\, e^{-i(\pi/L)} \\ \frac{25}{8} i \sin\left(\frac{\pi}{L}\right)(e^{i(2\pi/L)} - 1), & \frac{25}{8}\Delta t i \sin\left(\frac{\pi}{L}\right) - \frac{9}{16}\cos\left(\frac{\pi}{L}\right) \end{pmatrix}. \tag{19}$$

The nonhermitean matrix V has the eigenvalues

$$\begin{aligned} \lambda_{1,2} = {} & C_1 \cos\left(\frac{\pi}{L}\right) + C_2 i\,\Delta t \sin\left(\frac{\pi}{L}\right) \\ & \pm \frac{1}{2}\left(\left(C_3 \cos\left(\frac{\pi}{L}\right) + C_4 i\,\Delta t \sin\left(\frac{\pi}{L}\right)\right)^2 + C_5(0.25 - \Delta t^2)\sin^2\left(\frac{\pi}{L}\right)\right)^{1/2} \end{aligned} \tag{20}$$

with $C_1 = 7/32$, $C_2 = 41/16$, $C_3 = 25/16$, $C_4 = -9/8$, and $C_5 = 25/2$.

The stability condition for the solution of Eq. (10) is $\Delta t < 0.5\Delta x$ in the first-degree case and $\Delta t < 0.24\Delta X$ in the second-degree case.

In the second-degree case the two eigenvalues represent two scales with different phase velocities c and damping factors A_{n+1}/A_n. The comparison with the exact values $C_0 = -1$ and $A_{n+1}^0/A_n^{\,0} = 1$ is shown in Figs. 3 and 4.

The values of the relative velocity for the first-degree method (Fig. 3c) are comparable to those of the Eliassen grid (see diagrams in [1]). The values of A_{n+1}/A_n (Fig. 3d) reflect an excessive damping even of the longer waves. This makes an application to practical problems difficult. The first-degree method thus shows features similar to the one-sided difference scheme discussed in [11].

The second-degree method (Figs. 3a and 3b) gives a considerable improvement when compared with the Eliassen grid or the first-degree method. For a special

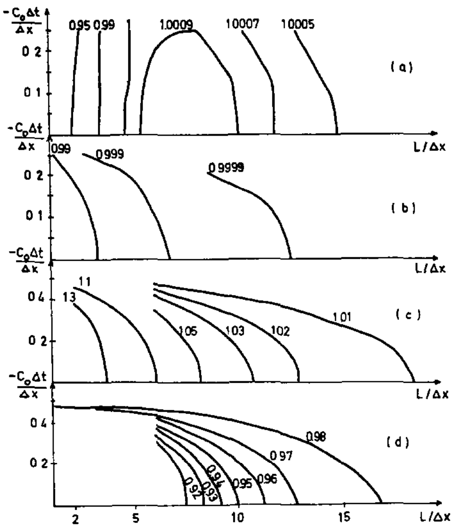

FIG. 3. Relative phase velocity and damping factors for first and second-degree method. (a) Relative phase velocity C/C_0 of second-degree method. (b) Damping factor A_{n+1}/A_n of second-degree method. (c) Relative phase velocity C/C_0 of first-degree method. (d) Damping factor A_{n+1}/A_n of first-degree method.

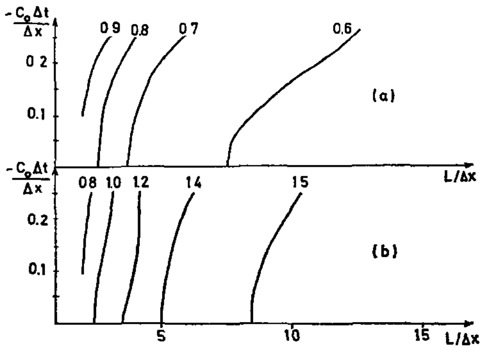

FIG. 4. Relative phase velocity and damping factor corresponding to the second eigenvalue of the second-degree method. (a) Damping factor A_{n+1}/A_n. (b) Relative phase velocity C/C_0.

value of Δt, relative phase velocities are given in [2] for several more advanced schemes. For this special Δt the results from the second-degree method are comparable to those obtained from a truncated Fourier scheme. For example, the relative phase velocity errors are smaller than 1/1000 for $L > 8\Delta x$ in the truncated Fourier case, and for $L > 4\Delta x$ in the second-degree case.

When comparing the damping factors from Fig. 3b with those of other dissipative schemes, such as the Euler backward method (see diagram in [1]), one finds a considerable improvement. On the other hand, there are many undissipative schemes. In nonlinear cases with realistic initial values these schemes require explicit dissipation terms to maintain nonlinear stability. The size of these terms will depend on the situation being considered. The dissipation thus introduced should be compared with the artificial damping given by Fig. 3b.

The second eigenvalue is a special feature of the second-degree method. The corresponding phase velocities and damping factors are plotted in Fig. 4. This eigenvalue causes a strong damping of some parts of the field. The role of this eigenvalue is illustrated by Fig. 5. It shows the action of the spatial smoothing operation in the $L \to \infty$ case.

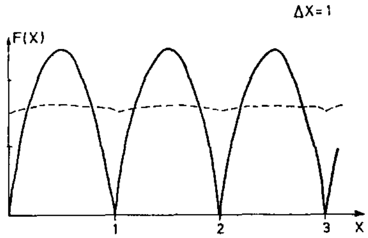

FIG. 5. Damping out of strong variability. ———, initial field; - - - -, field after six grid shiftings with $\Delta t = 0$.

4. Further Test Calculations

Solution of the Advection Equation with Positive Initial Values

Further solutions of Eq. (10) are now given for the positive initial values shown in Fig. 6a. The computation was done on a grid with 31 points and periodic boundary condition. The length of one period was therefore $L = 30\Delta X$. For the solution of the linear equation we consider t, x, and ϕ as dimensionless quantities and again choose $\Delta X = 1$.

The forecast time was always chosen in such a way that the exact solution of Eq. (10) would give the same diagram as the initial condition. The result for the first-degree method is given in Fig. 6e. It gives a function that is still positive, but the maximum has become much smaller and has spread out over the whole interval of definition. Looking at the results from the second-degree method, represented in Figs. 6b and 6c, one sees that the time translation acted simultaneously as a smoothing operation. The sharp peaks of the initial field are no longer present. The deformations of the field after time translation are not much different from

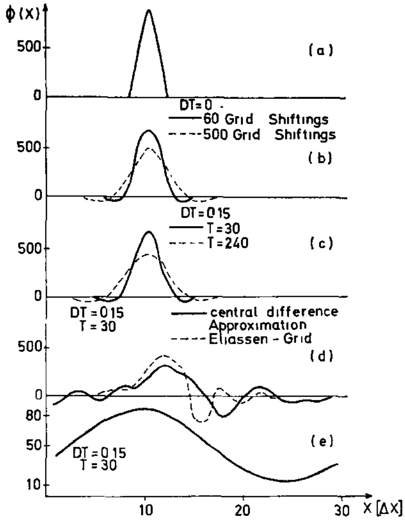

FIG. 6. Numerical solution of the advection equation with positive initial values. (a) initial condition; (b) grid shiftings without forecast in second-degree method; (c) time translation with second-degree method; (d) conventional methods (Eliassen grid) central difference approximation; (e) first-order method.

those effected by the grid shiftings alone. The field remained positive to a good approximation, while the spreading out is much less than in the first-degree case. Fig. 6d illustrates the common model errors associated with the numerical solution of the advection equation [2, 12]. For this purpose we used the centered difference approximation (solid)

$$\phi_\nu^{n+1} = \phi_\nu^{n-1} + \frac{2\Delta t}{2\Delta x} (\phi_{\nu+1}^n - \phi_{\nu-1}^n)$$

and the Eliassen grid (dashed)

$$\phi_{\nu+(1/2)}^{2n+1} = \phi_{\nu+(1/2)}^{2n-1} + \frac{2\Delta t}{\Delta x} (\phi_{\nu+1}^{2n} - \phi_\nu^{2n})$$

$$\phi_\nu^{2n} = \phi_\nu^{2n-2} + \frac{2\Delta t}{\Delta x} (\phi_{\nu+(1/2)}^{2n-1} - \phi_{\nu-(1/2)}^{2n-1}).$$

There are other methods of improving numerical advection (see for example [2, 8, 11, 12]).

A quantitative comparison to some of these methods was done using wave solutions of the advection equation.

Results of a Nonlinear Computation

To test the nonlinear numerical stability properties we consider a barotropic fluid (Eqs. (12)). The initial values are

$$U(x) = U_0 = 190.5 \text{ km/hr} \tag{21}$$

and

$$H(x) = H_0 = 2.0230296(381 \text{ km/hr})^2.$$

The same initial values were used in [13] in connection with orography, coriolis force, and periodic boundary condition. Here, we impose the rigid wall boundary conditions

$$U(0) = U(L) = 0 \qquad \frac{\partial}{\partial x} H(0) = \frac{\partial}{\partial x} H(L) = 0, \tag{22}$$

which pose a hard test for stability.

Eqs. (12) were solved by the second-degree method with $\Delta X = 381$ km, $\Delta t = 5$ min and $L = 29\Delta X$. Eqs. (22) were implemented only in the original grid (not in the shifted grid). This is done after the time translation step when the fields are represented as functions belonging to P_2. The formulas are given for the $X = L$ boundary (the time index is dropped):

$$\begin{aligned} \bar{U}_{59/2} &= -\bar{U}_{57/2} & \bar{H}_{59/2} &= \bar{H}_{57/2} \\ \bar{U}_{x,59/2} &= \bar{U}_{x,57/2} & \bar{H}_{x,59/2} &= -\bar{H}_{x,57/2} \\ \bar{U}_{xx,59/2} &= -\bar{U}_{xx,57/2} & \bar{H}_{xx,59/2} &= \bar{H}_{xx,57/2}. \end{aligned}$$

During the time development, which is shown in Fig. 7, different types of movement occur. After approximately 20 days the H field is nearly equal to the initial condition, while the U field is nearly zero.

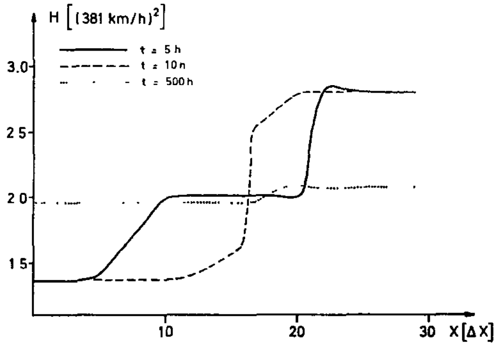

FIG. 7. The H-field at various times.

4. Conclusions

The second-degree method exibited a combination of small phase velocity errors, moderate spatial smoothing of the larger scales, and good nonlinear stability properties. A comparison of the first- and second-degree cases indicates that the solution converges quickly with the degree parameter. This might be a motivation to investigate the third-degree case.

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